Appendix 4.1

This contains material sometimes seen in lectures if there is enough time.

Example 4.30 *For* $Q \geq 1$ *let* $f_Q : [0,1] \to \mathbb{R}$ *be given by*

$$f_Q(x) = \begin{cases} 1 & \text{if } x = p/q, \text{ in reduced terms, } q \leq Q, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f_Q *is Riemann integrable over* $[0,1]$ *with integral equal to* 0*.*

Solution. Given $Q \geq 1$ the number of $0 \leq p/q \leq 1$, so $p \leq q \leq Q$ is $\leq Q^2$ (i.e. Q choices for q and $\leq Q$ choices for p).

Let $n \geq 1$ and $\mathcal{P}_n$ be the arithmetic partition of $[0,1]$ which gives n subintervals of width $1/n$.

In every subinterval there is an irrational number at which $f_Q(x)$ is 0, and thus $m_i = 0$ for all i. Hence $L(\mathcal{P}_n, f_Q) = 0$.

In at most Q^2 subintervals we will find a p/q in reduced form with $q \leq Q$. In such an interval $M_i = 1$. In all other intervals $f_Q(x) = 0$ throughout and so $M_i = 0$. Hence

$$U(\mathcal{P}_n, f_Q) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = \frac{1}{n} \sum_{i=1}^{n} M_i,$$

since $(x_i - x_{i-1}) = 1/n$ for all $1 \leq i \leq n$ in the arithmetic partition $\mathcal{P}_n$. Next, dropping the terms $M_i = 0$ we get

$$U(\mathcal{P}_n, f_Q) = \frac{1}{n} \sum_{\substack{i=1 \\ M_i=1}}^{n} 1 \leq \frac{Q^2}{n},$$

since $M_i = 1$ for at most Q^2 values of i.

Combining

$$0 = L(\mathcal{P}_n, f_Q) \leq \underline{\int_0^1} f_Q(x)\, dx \leq \overline{\int_0^1} f_Q(x)\, dx \leq U(\mathcal{P}_n, f_Q) \leq \frac{Q^2}{n}.$$

Let $n \to \infty$ to get

$$0 \leq \underline{\int_0^1} f_Q(x)\, dx \leq \overline{\int_0^1} f_Q(x)\, dx \leq 0, \quad \text{i.e.} \quad \underline{\int_0^1} f_Q(x)\, dx = \overline{\int_0^1} f_Q(x)\, dx = 0.$$

Therefore we have verified the definition that f_Q is Riemann integrable over $[0,1]$ with integral 0. ■

You should be able to check that for all $x \in [0,1]$ we have $\lim_{Q\to\infty} f_Q(x) = f(x)$, where is f is the non-integrable function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{otherwise.} \end{cases}$$

seen earlier. This is an unsatisfactory state of affairs, that the limit of a sequence of Riemann integrable functions is not Riemann integrable or that we **don't** have

$$\lim_{Q\to\infty} \int_0^1 f_Q(x)\,dx = \int_0^1 \lim_{Q\to\infty} f_Q(x)\,dx.$$

For the following result I said in the notes that it should not be hard for the interested student.

Theorem 4.31 *If f is monotonic on $[a,b]$ then f is Riemann integrable on $[a,b]$.*

Proof Assume f is increasing, the proof for decreasing is similar.

Let

$$\mathcal{P}_n = \left\{ a + \frac{b-a}{n} j : 0 \le j \le n \right\}$$

be the arithmetic partition of $[a,b]$ with n to be chosen. Then in the notes it was shown that

$$L(\mathcal{P}_n, f) = U(\mathcal{P}_n, f) + \frac{b-a}{n}\left(f(a) - f(b)\right). \tag{6}$$

Recall that for all $n \ge 1$,

$$L(\mathcal{P}_n, f) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(\mathcal{P}_n, f)$$

in which case

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le U(\mathcal{P}_n, f) - L(\mathcal{P}_n, f) = \frac{b-a}{n}\left(f(b) - f(a)\right)$$

by (6) . Let $n \to \infty$ to get

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq 0, \quad \text{i.e.} \quad \overline{\int_a^b} f = \underline{\int_a^b} f.$$

Hence we have verified the definition that f Riemann integrable on $[a, b]$. ■

Example 4.32 *Let*

$$f : [0, 1] \to \mathbb{R}, x \mapsto \frac{1}{1 + x^2}.$$

Prove that f is Riemann integrable over $[0, 1]$.

Solution f is a decreasing function on $[0, 1]$. ■

Example 4.33 *Let $f : [0, 1] \to \mathbb{R}$ be given by $f(0) = 0$ and, for $x \in (0, 1]$,*

$$f(x) = \frac{1}{n} \text{ where } n \text{ is the largest integer satisfying } x \leq \frac{1}{n}.$$

Draw the graph of f. Show that f is Riemann integrable on $[0, 1]$. *How many discontinuities does this function, f, have?*

If $F(x) = \int_0^x f(t)\, dt$ is $F'(x) = f(x)$ for all $x \in (0, 1)$?

Solution f is an increasing function on $[0, 1]$ so it is integrable. It has a countable infinity of discontinuities, showing that it is possible to integrate a function with so many discontinuities.

You should be able to show that $F'(x)$ does not exist when x is of the form $1/n$ for some $n \geq 1$. Therefore at these points we cannot have $F'(x)$ equals anything, let alone $f(x)$. ■

Appendix 4.2

The observant student would have seen that we assumed the following result in the proof of the Fundamental Theorem of Calculus.

Theorem 4.34 *Assume that the bounded functions f and g are Riemann integrable on $[a, b]$. Then*

(i) Linearity $f + g$ is Riemann integrable on $[a, b]$ with

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

(ii) Additive Property. For $a < c < b$, the function f in integrable over the sub-intervals $[a, c]$ and $[c, b]$ with

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof not given. ■

The first result here is an extension of the first result in

Theorem 4.35 *Assume that the bounded functions f and g are Riemann integrable on $[a, b]$. Then*

(i) ***Sum Rule****: $f + g$ is Riemann integrable on $[a, b]$.*

(ii) ***Product Rule****: fg is Riemann integrable on $[a, b]$.*

(iii) ***Quotient Rule****: f/g is Riemann integrable on $[a, b]$ if there exists $C > 0$ such that $|g(x)| \geq C$ for all $x \in [a, b]$.*

(iv) the function $|f|$, defined by $|f|(x) = |f(x)|$ for all $x \in [a, b]$, is Riemann integrable over $[a, b]$.

Proof not given. ■

Of course there are no simple relationships between the integrals of fg and f/g with f and g. While for $|f|$ we saw (without proof) in the notes that

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Appendix 4.3 Integration as the inverse of differentiation.

Though the primitive of a continuous function need not be unique, because of a possible constant, the value of definite integral is the difference between the primitive evaluated at two points, when the unknown constant vanishes. Thus

Corollary 4.36 ***Definite integral*** *If g is continuous on $[a, b]$ and G is a primitive for g then*

$$\int_y^x g(t)\, dt = G(x) - G(y) = [G(t)]_y^x \tag{7}$$

for all $a \leq y < x \leq b$.

Example 4.37 *Prove that $\ln x$, defined earlier as the inverse of e^x, satisfies*

$$\ln x = \int_1^x \frac{dt}{t}$$

for all $x > 0$. This is often taken as the definition of the natural logarithm.

Solution On Question sheet. ■

The Fundamental Theorem tells us how to differentiate an integral.

Example 4.38 *Let*

$$G(x) = \int_{x^2}^{x^3} e^t \cos t dt.$$

Calculate $G'(t)$.

Solution. Let

$$F(y) = \int_1^y e^t \cos t dt,$$

so $G(x) = F(x^3) - F(x^2)$. Since $e^t \cos t$ is continuous we have, by the Fundamental Theorem, that F is differentiable and $F'(y) = e^y \cos y$. Thus, by the composite rule for differentiation,

$$\begin{aligned} G'(x) &= 3x^2 F'\left(x^3\right) - 2xF'\left(x^2\right) \\ &= 3x^2 e^{x^3} \cos\left(x^3\right) - 2xe^{x^2} \cos\left(x^2\right). \end{aligned}$$

■

The next result follows from the simple observation that if F and G are primitives for f and g respectively then FG is a primitive for $Fg + fG$, by the Product Rule for differentiation. Then (7) gives

$$\int_y^x (F(t)\, g(t) + f(t)\, G(t))\, dt = [F(t)\, G(t)]_y^x ,$$

i.e.

$$\int_y^x (F(t)\, G'(t) + F'(t)\, G(t))\, dt = [F(t)\, G(t)]_y^x .$$

It is more normal to write these functions as lower case (so f is **not** F', but is rather **replacing** F, similarly with g). But be careful, since f and g are replacing primitives they must share the properties of primitive functions, i.e. differentiable with continuous derivatives.

Theorem 4.39 ***Integration by parts***. *Assume that f and g have continuous derivatives on $[a, b]$. Then*

$$\int_y^x f(t)\, g'(t)\, dt = [f(t)\, g(t)]_y^x - \int_y^x f'(t)\, g(t)\, dt.$$

Proof completed. ■

As an example of the use of this, repeated application leads to the integral form of the error term in Taylor's Theorem.

Example 4.40 *(Cauchy) Prove that if $f^{(n+1)}$ is continuous on $[a, b]$ then*

$$\int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt = R_{n,a} f(x)$$

for all $a \le x \le b$, where $R_{n,a}$ is the remainder term for the n-th Taylor Series.

Proof See Question Sheet. ■

Applications Already used in previous Part 3 of this course to justify

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots .$$

Above the Product Rule for differentiation led to a rule for integration. Would the Chain or Composite Rule for differentiation lead to anything for integrals? Assume that g is continuous on $[a,b]$ and differentiable on (a,b). Thus $g:[a,b]\to I$ for some set I. Assume that F is a primitive for f on I (for which it suffices that f is continuous on $g([a,b])$.) Then the Composite Rule for differentiation gives

$$(F\circ g)'(t) = (F'\circ g(t))\, g'(t)$$

on (a,b). Then (7) gives

$$\begin{aligned}
\int_a^b (F'\circ g(t))\, g'(t)\, dt &= [F\circ g(t)]_a^b \\
&= F(g(b)) - F(g(a)) \\
&= [F]_{g(a)}^{g(b)} \\
&= \int_{g(a)}^{g(b)} F'(x)\, dx,
\end{aligned}$$

using (7) again in the last step. This gives

Theorem 4.41 ***Change of Variable or Integration by Substitution*** *Assume that g is continuous on $[a,b]$ and differentiable on (a,b).and f is continuous on $g([a,b])$. Then*

$$\int_{g(a)}^{g(b)} F'(x)\, dx = \int_a^b (F'\circ g(t))\, g'(t)\, dt$$

where $a = g(\alpha)$ and $b = g(\beta)$.

Proof completed. ■

Note This result is often written as

$$\int_\alpha^\beta F'(x)\, dx = \int_{g^{-1}(\alpha)}^{g^{-1}(\beta)} (F'\circ g(t))\, g'(t)\, dt,$$

but this requires the existence of the inverse of g. Since g is continuous we will require g to be strictly monotonic.

Example 4.42 *Prove that*

$$\tanh x = \int_0^x \frac{dt}{\cosh^2 t}$$

for $x\in\mathbb{R}$.

Solution From the definition of $\cosh x$ we have

$$\int_0^x \frac{dt}{\cosh^2 t} = 4\int_0^x \frac{dt}{(e^t + e^{-t})^2}.$$

Make the change of variable $y = e^t$ to get

$$\begin{aligned}
\int_0^x \frac{dt}{(e^t + e^{-t})^2} &= 4\int_1^{e^x} \frac{dy}{y\,(y + y^{-1})^2} = 4\int_1^{e^x} \frac{ydy}{(y^2+1)^2} \\
&= \left[-\frac{2}{y^2+1}\right]_1^{e^x} = 1 - \frac{2}{e^{2x}+1} \\
&= \frac{e^{2x}-1}{e^{2x}+1} = \tanh x.
\end{aligned}$$

■

Example 4.43 *With $\ln x$ expressed as an integral, as we saw in an earlier example, prove that for all $a, b > 0$ we have $\ln ab = \ln a + \ln b$.*

Solution on Question sheet. ■

Appendix 4.4 Improper Integrals

We have only defined the integral for closed intervals and functions bounded on such intervals. We finish with a list of definitions that try to extend the situations in which our definitions are meaningful.

Definition 4.44 *If $f : [a, \infty) \to \mathbb{R}$ is Riemann integrable on every interval $[a, b]$ and*

$$\lim_{t \to \infty} \int_a^t f(x)\, dx$$

makes sense with the limit existing, we define this limit to be $\int_a^\infty f(x)\, dx$ and we say that the integral ***converges****. Otherwise is* ***diverges****.*

Similarly for $\int_{-\infty}^b f(x)\, dx$.

Example 4.45 *Show that*

$$\int_1^\infty \frac{1}{x^\alpha} dx$$

converges if, and only if, $\alpha > 1$.

Solution on Question sheet. ■

Definition 4.46 *We define*

$$\int_{-\infty}^\infty f(x)\, dx \quad as \quad \int_{-\infty}^0 f(x)\, dx + \int_0^\infty f(x)\, dx,$$

and it converges only if both of these do so separately.

Definition 4.47 *(Gauss, 1812) If f is defined on $(a, b]$ and $\lim_{\alpha \to a} \int_\alpha^b f(x)\, dx$ makes sense and the limit exists, then we define the limit to be $\int_a^b f(x)\, dx$ and say that the integral* ***converges****. Otherwise is* ***diverges****.*

Similarly for a function defined on $[a, b)$.

Example 4.48 *For what α does*

$$\int_0^1 \frac{dx}{x^\alpha}$$

converge?

Solution on Question sheet. ■

Definition 4.49 *Assume that f is not defined at c in $[a, b]$ but, for all $\eta > 0$, is bounded in $[a, c-\eta] \cup [c+\eta, b]$. Then the **Cauchy Principal Value Integral** is defined to be*

$$P.V \int_a^b f(x)\,dx = \lim_{\eta \to 0} \left(\int_a^{c-\eta} f(x)\,dx + \int_{c+\eta}^b f(x)\,dx \right),$$

provided that this limit exists.

Appendix 4.5

The previous appendix contained material that sometimes is covered in lectures, if there is enough time. This appendix covers material that is either omitted from lectures due to lack of time (often left to students to do) or is background to and expands on the lectures.

1. Riemann gave his definition of an integral in 1854. Here we have given an approach due to Darboux from 1875. The upper and lower sums above should strictly be called Upper and Lower Darboux sums. They differ slightly from the Upper and Lower Riemann sums that you might find in alternative accounts of integration. But be careful! In a book by Strichartz the Darboux sums are called Riemann sums while, what I would call Riemann sums, are called Cauchy sums. Very confusing!

2. Earlier in the course I termed the function $f:[0,1]\to\mathbb{R}$ given by

$$f(x)=\begin{cases}1 & \text{if } x \text{ rational}\\ 0 & \text{if } x \text{ irrational}\end{cases}$$

pathological. It is not though, in the scheme of things, a complicated function (imagine what a continuous nowhere-differentiable function might look like). In fact it is zero except for at a countable number of points. So it is a weakness of the theory of Riemann integration that we can't integrate this function. In a third year course a theory of integration due to Lebesgue, from 1902, is studied. With Lebesgue integration this function can be integrated. Can you guess at the value of the integral?

3. Let $\mathcal{P}=\{x_i : 0\le i\le n\}$ and $\mathcal{D}$ be a refinement of $\mathcal{P}$. It was left for the student to prove that $U(\mathcal{D},f)\le U(\mathcal{P},f)$.

The Upper sum for $\mathcal{P}$ is $U(\mathcal{P},f)=\sum_{i=1}^{n}M_i\left(x_i-x_{i-1}\right)$ Choose any $y\in\mathcal{D}\setminus\mathcal{P}$. Thus there exists $1\le j\le n$ such that $x_{j-1}<y<x_j$. Then

$$U(\mathcal{P}\cup\{y\},f)=\sum_{i\neq j}M_i\left(x_i-x_{i-1}\right)+\underset{[x_{j-1},y]}{\text{lub}}f\ \left(y-x_{j-1}\right)+\underset{[y,x_j]}{\text{lub}}f\left(x_j-y\right) \tag{8}$$

We then use the facts that $[x_{j-1},y]$, $[y,x_j]\subseteq[x_{j-1},x_i]$ and there is a chance that f takes *larger* values when we extend the interval. Thus

$$\underset{[x_{j-1},y]}{\text{lub}}f\ \le\ \underset{[x_{j-1},x_j]}{\text{lub}}f\ =M_j\quad\text{and}\quad \underset{[y,x_j]}{\text{lub}}f\ \le\ \underset{[x_{j-1},x_j]}{\text{lub}}f\ =M_j.$$

Hence RHS(8) is

$$\begin{aligned}
&\leq \sum_{i\neq j} M_i\left(x_i - x_{i-1}\right) + M_j\left(y - x_{j-1}\right) + M_j\left(x_j - y\right)\\
&= \sum_{i\neq j} M_i\left(x_i - x_{i-1}\right) + M_j\left\{\left(y - x_{j-1}\right) + \left(x_j - y\right)\right\}\\
&= \sum_{i\neq j} M_i\left(x_i - x_{i-1}\right) + M_j\left(x_j - x_{j-1}\right) = U\left(\mathcal{P}, f\right).
\end{aligned}$$

That is, $U(\mathcal{P}\cup\{y\}, f) \leq U(\mathcal{P}, f)$. Continue adding in points from $\mathcal{D}\setminus\mathcal{P}$, to get $U(\mathcal{D}, f) \leq U(\mathcal{P}, f)$.

4. For the statement of another criterion of integration define $\partial\mathcal{P}$, of a partition $\mathcal{P} = \{a = x_0 < x_1 < .... < x_n = b\}$, to be $\partial\mathcal{P} = \max_{1\leq i\leq n}\left|x_i - x_{i-1}\right|$, i.e. the maximum partition length.

Theorem 4.50 ***(Du Bois-Reymond 1875, Darboux 1875)*** *A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann Integrable if, and only if,*

$$\forall\varepsilon > 0, \exists\,\delta > 0, \forall\mathcal{P} \in D_\delta, U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon.$$

Here D_δ is the set of all partitions with $\partial\mathcal{P} < \delta$.

Proof Not given. ■

For an application of this last result assume $f : [0,1] \to \mathbb{R}$ is continuous. We know then that it is bounded and Riemann integrable. Let $\mathcal{P}_n$ be the sequence of arithmetic partitions of $[0,1]$. Since $\partial\mathcal{P}_n = 1/n$, Du Bois-Reymond and Darboux's Theorem shows that both $U(\mathcal{P}_n, f)$ and $L(\mathcal{P}_n, f) \to \int_0^1 f(x)\,dx$ as $n \to \infty$. Yet on each sub-interval $[x_{i-1}, x_i] = [(i-1)/n, i/n]$, of the partition $\mathcal{P}_n$ we have $m_i \leq f\left(i/n\right) \leq M_i$. Hence

$$L(\mathcal{P}_n, f) \leq \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) \leq U\left(\mathcal{P}_n, f\right).$$

Thus by the sandwich rule we deduce that

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) \to \int_0^1 f(x)\,dx$$

as $n \to \infty$. This was the *Integral Approximation Rule*, Theorem 1.48, of MATH10242.

5. The proof of the result that continuous functions are integrable depends on a property of continuous functions that we have not discussed. This property is that if $f : [a, b] \to \mathbb{R}$ is continuous then, since the domain $[a, b]$ is closed and bounded, we have

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in [a, b], |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon. \tag{9}$$

Try to understand why this statement is different to the definition of continuity. A function that satisfies (9) is said to be *uniformly continuous.*

6. The result that $\int_1^\infty \frac{1}{x^\alpha} dx$ converges if, and only if, $\alpha > 1$ is reminiscent of a result in course MATH10242 concerning those k for which $\sum_{r=1}^\infty \frac{1}{n^k}$ converges. But then there is often a connection between the series $\sum_1^\infty f(n)$ and integral $\int_1^\infty f(t)\, dt$. For instance, if $f : [1, \infty) \to \mathbb{R}$ is a positive decreasing integrable function then the series and integral either both diverge or both converge. This was Theorem 2.16, the Integral Test of MATH10242.

7. The following is an example given without solution in the notes.

Example 4.51 *Define* f *on* $[0, 2]$ *by*

$$f(x) = \begin{cases} 0 & \textit{if } 0 \le x \le 1 \\ 1 & \textit{if } 1 < x \le 2. \end{cases}$$

What is $F(x)$? *What is* $F'(1)$?

Solution

$$F(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1 \\ x - 1 & \text{if } 1 < x \le 2. \end{cases}$$

$$\lim_{x \to 1-} \frac{F(x) - F(1)}{x - 1} = 0 \quad \text{while} \quad \lim_{x \to 1+} \frac{F(x) - F(1)}{x - 1} = 1.$$

Thus $F'(1)$ does not exist. ■